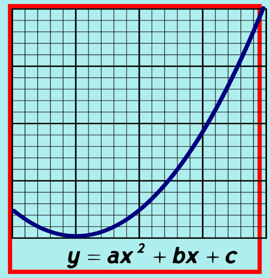
Math 25

Fall 2017

Lecture 9



Sequences & Series

Recall what a function is:

It is a relation that for every input values, there is only one output Value.

Notation $f(x)$

x input → Domain

$f(x)$ output → Range

Ex: $F(x) = 3x - 2$

$f(1) = 3(1) - 2 = 1$

$f(2) = 3(2) - 2 = 4$

$f(3) = 3(3) - 2 = 7$

$\vdots \qquad \vdots$

If we list the output

$1, \ 4, \ 7, \ \ldots$ This is a sequence.

$f(x) = x^2$

$f(1) = 1^2 = 1$

$f(2) = 2^2 = 4$

$f(3) = 3^2 = 9$

$\vdots \qquad \vdots$

as a sequence $\Rightarrow 1, \ 4, \ 9, \ \ldots$

So Sequence is a special function such that its domain is the set of natural numbers $1, 2, 3, \ldots$

Traditionally speaking, Sequence is in the form of

$\{a_n\}$ where $a_n = f(n)$,

a_1 First element $= f(1)$

a_2 Second element $= f(2)$

a_3 third element $= f(3)$

Find the first 4 terms of the sequence

$\{n^2-2n\}$

$a_n = n^2 - 2n$

$a_1 = 1^2 - 2(1) = -1$

$a_2 = 2^2 - 2(2) = 0$

$a_3 = 3^2 - 2(3) = 3$

$a_4 = 4^2 - 2(4) = 8$

$\{-1, 0, 3, 8, \ldots\}$

$n!$ n-Factorial

$n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

$0! = 1$ By definition

Find the first 5 terms of the sequence

$\{n! - 1\}$

$a_n = n! - 1$

$a_1 = 1! - 1 = 1 - 1 = 0$

$a_2 = 2! - 1 = 2 - 1 = 1$

$a_3 = 3! - 1 = 6 - 1 = 5$

$a_4 = 4! - 1$
$= 24 - 1 = 23$

$a_5 = 5! - 1 = 120 - 1$
$= 119$

$\{0, 1, 5, 23, 119, \ldots\}$

$a_1 = 1 \quad a_2 = 1, \quad a_{n+2} = a_{n+1} + a_n$

Any term is the sum of two previous terms

$1, 1, 2, 3, 5, 8, 13, \ldots$

$a_{n+1} = n\, a_n, \quad a_1 = 2$

$a_2 = 1 a_1 \qquad a_2 = 1 \cdot 2 = 2$

$a_3 = 2 a_2 \qquad a_3 = 2 \cdot 2 = 4$

$a_4 = 3 a_3 \qquad a_4 = 3 \cdot 4 = 12$

$a_5 = 4 a_4 = 48$

$\vdots$

Find the first 4 terms of

$$\left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty}$$

$$\left\{ \frac{1}{1+1}, \frac{2}{2+1}, \frac{3}{3+1}, \frac{4}{4+1}, \ldots \right\}$$

$$= \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots \right\}$$

$$= \{ .5, .\overline{6}, .75, .8, \ldots$$

what do you think this sequence approaches?

$\to 1$

$\sum$ Summation

$\sum_{n=1}^{K}$ Summation

$K \leftarrow$ Ends

$n=1 \leftarrow$ Starts

$$\sum_{n=1}^{4} n^2$$

$$= 1^2 + 2^2 + 3^2 + 4^2 = 30$$

$$\sum_{n=1}^{4}(3n-1) = \underset{n=1}{(3\cdot1-1)} + \underset{n=2}{(3\cdot2-1)} + \underset{n=3}{(3\cdot3-1)} + \underset{n=4}{(3\cdot4-1)}$$

$$= 2 + 5 + 8 + 11 = 26$$

These are examples of Series.

Series are the sum of terms of a Sequence.

$$\sum_{n=1}^{K} a_n$$

Find $\sum_{n=1}^{4}(n^3-n)$

$$= (1^3-1) + (2^3-2) + (3^3-3) + (4^3-4)$$

$$= 0 + 6 + 24 + 60$$

$$= \boxed{90}$$

Find $\sum_{n=1}^{99} \frac{1}{n^2+n}$

Do Partial fraction decomposition on $\frac{1}{n^2+n}$.

$$\frac{1}{n^2+n} = \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

we should get $1 = A(n+1) + Bn$

If $n=0 \to A=1$

If $n=-1 \to B=-1$

$$\Rightarrow \frac{1}{n^2+n} = \frac{1}{n} - \frac{1}{n+1}$$

So $\sum_{n=1}^{99} \frac{1}{n^2+n} = \sum_{n=1}^{99} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$$= \underbrace{\left(\frac{1}{1} - \cancel{\frac{1}{2}}\right)}_{n=1} + \underbrace{\left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right)}_{n=2} + \underbrace{\left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right)}_{n=3} \cancel{+} \cdots \cancel{+} \underbrace{\left(\cancel{\frac{1}{99}} - \frac{1}{100}\right)}_{n=99}$$

$$= 1 - \frac{1}{100} = \frac{100-1}{100} = \boxed{\frac{99}{100}}$$

what do You think about

$\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ approaches? $\to 1$

Back to Sequences

1) Arithmetic $\Rightarrow a_1, a_1+d, a_1+2d+a_1+3d, \ldots$

2) Geometric $\Rightarrow a_1, a_1 r, a_1 r^2, a_1 r^3, \ldots$

Arithmetic Sequence

a_1 First Term

a_n nth term

d Common Difference

n # of terms.

S_n Sum of the first n terms

Given $a_1 = 5$, $d = 3$, Arithmetic sequence

Find

1) the first 4 terms

$5, 5+3, 5+3+3, 5+3+3+3$

$= 5, 5+1\cdot 3, 5+2\cdot 3, 5+3\cdot 3$

$= 5, 8, 11, 14, \ldots$

2) S_4

$S_4 = 5+8+11+14 = 38$

a_1

$a_2 = a_1 + d$

$a_3 = a_1 + d + d = a_1 + 2d$

$a_4 = a_1 + 2d + d = a_1 + 3d$

$a_5 = a_1 + 3d + d = a_1 + 4d$

$\vdots$

$a_n = a_1 + (n-1)d$

For last example $a_{100} = a_1 + 99d$

$\to a_{100} = 5 + 99 \cdot 3$

$\boxed{a_{100} = 302}$

Consider the sequence below

2, 7, 12, 17, 22, ...

$a_1 = 2$, $d = 5$

find a_{20}.

$a_n = a_1 + (n-1)d$

$a_{20} = 2 + (20-1) \cdot 5$

$= \boxed{97}$

find the sum below

$1 + 2 + 3 + 4 + \cdots\cdots + 98 + 99 + 100.$

$$\begin{array}{l} 1 + 2 + 3 + \cdots + 98 + 99 + 100 \\ 100 + 99 + 98 + \cdots + 3 + 2 + 1 \\ \hline 101 + 101 + 101 + \cdots + 101 + 101 + 101 \end{array}$$

So we have 100 of these 101's.

Since we doubled up, we divide by 2

$$\frac{100(101)}{2} = 50(101)$$
$$= 5050$$

Karl Gauss

$$\begin{array}{l} a_1 + a_1+d + a_1+2d + \cdots + a_1+(n-1)d \\ a_1+(n-1)d + a_1+(n-2)d + \cdots + a_1 \end{array}$$

$$\underbrace{2a_1+(n-1)d + 2a_1+(n-1)d + \cdots + 2a_1+(n-1)d}_{n \text{ of these}}$$

$$S_n = \frac{n}{2}\left[2a_1+(n-1)d\right] \qquad S_n = \frac{n}{2}\left[a_1+a_n\right]$$

Sum of the first n terms

$20 + 24 + 28 + 32 + \cdots\cdots$

$a_1 = 20$

$d = 4$

find a_{25}

we know

$a_n = a_1 + (n-1)d$

$a_n = 20 + (n-1)\cdot 4$

$a_n = 20 + 4n - 4$

$a_n = 4n + 16$

$a_{25} = 4(25) + 16$

$\boxed{a_{25} = 116}$

find S_{25}

we know

$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$

So $S_n = \frac{n}{2}\left[2\cdot 20 + (n-1)4\right]$

$= \frac{n}{2}\left[40 + 4n - 4\right]$

$= \frac{n}{2}\left[4n + 36\right]$

$= \frac{n}{2}\cdot 4\left[n + 9\right]$

$S_n = 2n\left[n + 9\right]$

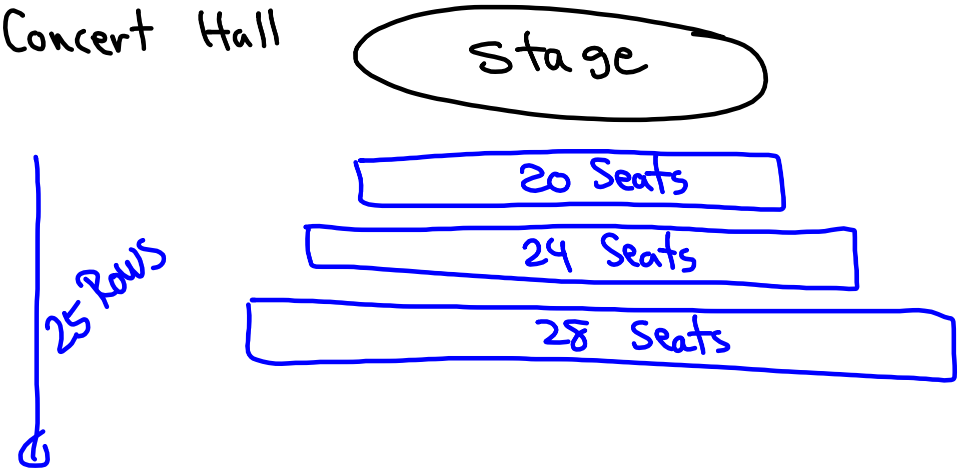
$S_n = 2n\left[n + 9\right]$

$S_{25} = 2\cdot 25\left[25 + 9\right]$

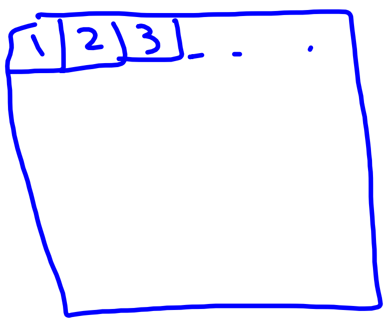
$= 50(34)$

$\boxed{S_{25} = 1700}$

Concert Hall



How many seats in total? 1700 Seats.



8 × 8

1 + 2 + 4 + 16 + . . .

For 64 Years.

Making a new Year resolution

Jan 1 → 1 ¢

Jan 2 → 2 ¢

Jan 3 → 4 ¢

⋮

Jan 31 →

How much $ do You have by Feb. 1?

Geometric Sequences

3, 6, 12, 24, 48, - - - - -

$a_1 = 3$, $r = 2$

common ratio

$3, \ 3 \cdot 2, \ 3 \cdot 2^2, \ 3 \cdot 2^3, \ 3 \cdot 2^4, \ 3 \cdot 2^5, \ldots$

find the 10th term $\Rightarrow a_{10}$

$$a_{10} = 3 \cdot 2^{10-1} = 3 \cdot 2^9 = \boxed{1536}$$

$$a_1, \ a_1 r, \ a_1 r^2, \ a_1 r^3, \ldots \underbrace{a_1 r^{n-1}}_{\text{nth term}}$$

use a geometric sequence with $a_1 = 1024$, and $r = \frac{1}{2}$
write the first 4 terms.

1024, 512, 256, 128

find a_8

$$a_8 = a_1 \cdot r^7$$
$$= 1024 \cdot \left(\tfrac{1}{2}\right)^7 = \boxed{8}$$

$S_n =$ Sum of the first n terms

For Arithmetic Sequence

$$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

For Geometric Sequence

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

So Save 1¢ on Jan 1, double it for Jan 2 double that for Jan 3, How much in total be end of Jan?

$a_1 = 1$

$r = 2$

$n = 31$

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$S_{31} = \frac{1(1-2^{31})}{1-2}$$

$$= \frac{1-2^{31}}{-1} = 2^{31} - 1$$

$S_{31} = 2\,147\,483\,647$ ¢

\$ 21,474,836.47